Unsteady Surface-Element Method Applied to Mixed-Boundary Conditions with Accuracy Study

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This paper presents the application of the unsteady surface-element (USE) method to the heat condition problem in a semi-infinite body with mixed boundary conditions of a step change of the surface temperature over an infinite strip and insulated elsewhere. The accuracy of the method is studied by comparing the results with those obtained from the approximate analytical solution (using the Laplace transform technique), and also by considering different sizes of elements and the time steps used in the numerical computations. The results show that a very high accuracy is attainable with a relatively small number of surface elements. It is also found that the accuracy of results varies linearly with the sizes of element and time step.

Nomenclature

a	= half of the width of the contact area
A_c	= contact area
A_j	= area of the surface element j
erfc(.)	= complementary error function
$\boldsymbol{F}$	= vector defined by Eq. (7c)
k	= thermal conductivity
M	= time index
N	= number of surface elements
q	= heat flux
q_{cL}	= centerline heat flux
$\bar{q}$	= area-averaged interface heat flux
$\boldsymbol{q}_M$	= heat flux vector at time t_M
t	= time
t^+	= dimensionless time
T	= termperature
T_c	= contact-area temperature
T_0	= initial temperature
x, z	= Cartesian coordinates
x^+, z^+	= Dimensionless x, z defined by (Eq. 16)
α	= thermal diffusivity
γ	= Euler's constant
λ	= dummy variable
ϕ	= temperature rise for a unit heat flux
ϕ_{kji}	= temperature rise at element k due to unit heat flux over element j at time t_i
$\vec{\Phi}_i$	= influence matrix at time t_i

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Introduction

THE unsteady surface-element (USE) method is a powerful analytical and numerical technique for solving certain transient heat-transfer problems. The method is most suitable for calculating interface temperatures and heat fluxes for the geometries connected over a relatively small portion of their surface boundaries. It is applicable to homogeneous and composite geometries with perfect or imperfect contacts.

There are two different approaches for the USE method. One is the single node approach[1] that uses the Laplace transform (approximate analytical solution), and the other is the multinode approach[2] that is more general and can be applied to a variety of problems (numerical solution). Both approaches use Duhamel's integral and consequently can only be applied to problems with the linear-differential equations.

In this paper, the multinode unsteady surface-element (MUSE) method is employed to obtain the transient thermal response of a semi-infinite body with the mixed boundary conditions of a step change of the surface temperature over an infinite strip and insulated elsewhere. See Fig. 1. This problem is similar to that of two semi-infinite bodies with identical properties having different initial temperatures brought suddenly into perfect thermal contact over an infinite strip and insulated over the rest of the contacting area.

The solution has application in problems involving electronic cooling, strip welding, fins, and thermal contact conductance. To the authors' knowledge, there is no analytical

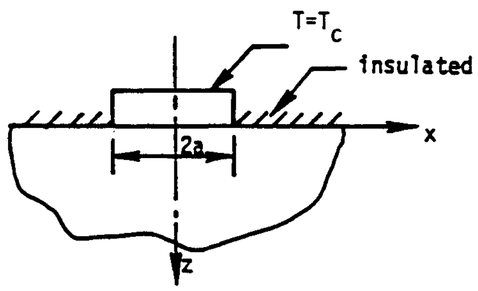

Fig. 1 Geometry of semi-infinite body with the step change of surface temperature over an infinite strip.

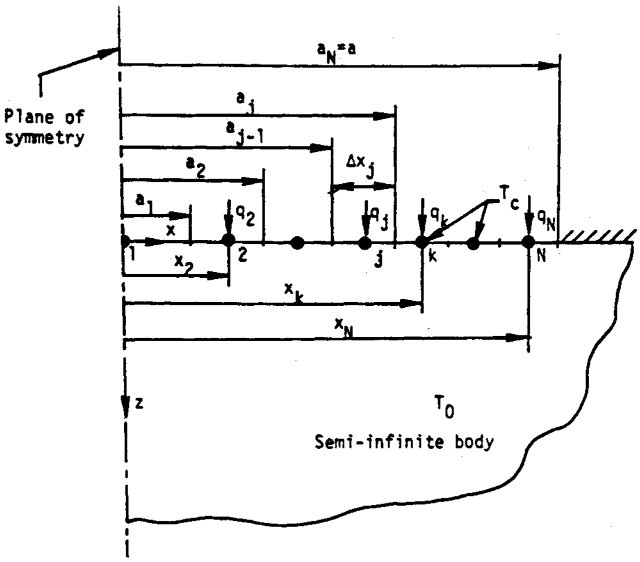

Fig. 2 Possible distribution of surface elements over the surface boundary.

solution to the foregoing problem currently available in the open literature. Sadhel[3] has examined the related problem of two semi-infinite bodies having perfect contact over a series of equally spaced infinite strips. The regions between these strips were insulated. By considering the planes of symmetry between the strips, Sadhel solved the problem for large times by utilizing the Laplace transform technique. However, his solution is not valid for the situation in which there is a small fraction of the interface in contact and consequently cannot be applied to the problem of two semi-infinite bodies with a single strip contact.

Two types of kernels (influence functions or fundamental solutions) can be employed in the USE method: a temperature-based kernel and a heat-flux-based kernel. The method requires that these influence functions be known for the basic geometries under consideration. For many geometries, the influence functions are known or can efficiently be obtained by the analytical method or the use of Green's functions.[4] Although both types of kernels can be employed by the USE method, only the heat-flux-based kernels are used in the example problem given in this paper.

Statement of the Problem

The geometry being considered is shown in Fig. 1. A semi-infinite body is subjected to a uniform step temperature change T_c over an infinite strip with width $2a$ on its surface. The rest of the surface is insulated. The body is initially at a uniform temperature of T_0, and the thermal properties are assumed to be independent of temperature. The describing equations are

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{1}$$

$$T = T_0 \quad \text{for} \quad t = 0; \qquad |x| \geq 0; \qquad z \geq 0 \tag{2}$$

$$T = T_c \quad \text{for} \quad t > 0; \qquad -a \leq x \leq a; \qquad z = 0 \tag{3a}$$

$$\frac{\partial T}{\partial z} = 0 \quad \text{for} \quad t > 0; \qquad |x| > a; \qquad z = 0 \tag{3b}$$

$$T = T_0 \quad \text{for} \quad t > 0 \quad \text{as} \quad x \to \pm\infty \quad \text{and} \quad z \to \infty \tag{4}$$

Surface-Element Solution (Multinode)

In the heat-flux-based MUSE method, the part of the surface boundaries with nonzero values of that flux is divided into N finite surface elements (not necessarily equal) with each of these elements having inner and outer dimensions denoted by a_j and a_{j-1}, as shown in Fig. 2. In general, the heat flux and temperature vary across the surface boundary, but they are assumed to be uniform over each surface element. If we use Duhamel's integral, the temperature at location (x, z) and time t due to the effect of N heat flux histories $[q_j(t), j = 1, N]$ can be given by

$$T(x, z, t) - T_0 = \sum_{j=0}^{N} \int_0^t q_j(\lambda) \frac{\partial \phi_j(x, z, t-\lambda)}{\partial t} \, d\lambda \tag{5}$$

where $\phi_j(x, z, t)$ is the temperature rise at location (x, z) and time t due to a unit step heat flux over element j at time zero. It represents the fundamental solution or influence function for the aforementioned geometry. For a point along the interface boundary ($z = 0$), Eq. (5) reduces to

$$T_k(t) - T_0 = \sum_{j=0}^{N} \int_0^t q_j(\lambda) \frac{\partial \phi_{kj}(t-\lambda)}{\partial t} \, d\lambda \quad \text{for } k = 1,2,\ldots,N \tag{6}$$

where $T_k(t)$ is the temperature of surface element k (both local and average values are considered in this work), and $\phi_{kj}(t)$ represents the temperature rise at surface element k due to a unit heat flux over element j. For the example problem in this paper, $T_k(t)$ is known and equal to T_c. Therefore, the set of integral equations given by Eq. (6) can be solved numerically for the elemental heat fluxes $q_j(t)$, $j = 1, N$ (the only unknowns). Once q_j is evaluated, the temperature at any point inside the body can be found by the direct integration of Eq. (5).

The time integrations in Eqs. (5) and (6) are performed directly by dividing the entire time domain into M equal small time intervals Δt, so that t_M represents the value of t at the end point of the Mth interval. With the uniform approximation of the heat flux histories $q_j(t)$ over each time interval, one can show that

$$\boldsymbol{q}_1 = \boldsymbol{\Phi}_1^{-1} \boldsymbol{T}_{c0}; \qquad T_{c0} = T_c - T_0 \tag{7a}$$

$$\boldsymbol{q}_M = M\boldsymbol{q}_1 - \boldsymbol{\Phi}_1^{-1} \boldsymbol{F}_M \qquad \text{for} \qquad M = 2, 3, \ldots \tag{7b}$$

$$\boldsymbol{F} = \sum_{i=1}^{M-1} \boldsymbol{\Phi}_{m+1-i} \boldsymbol{q}_i \tag{7c}$$

where $\boldsymbol{q}_i$ and $\boldsymbol{\Phi}_i$ represent the heat flux vector and the influence matrix at time t_i, respectively. The elements of the influence matrix are $\phi_{kj}(t_i)$.

It should be noted that because of the convolution behavior of Eq. (7c), the influence matrix needs to be evaluated at each time step and stored for later use. This can result in a large computational effort, particularly if the number of time steps (NTS) is large. However, it will be shown later that in most cases (except for very early times), a value of NTS between 20 and 40 can produce very accurate results.

Because of the symmetrical nature (about the x axis) of the example problem just described, the solution is required only for the region $x > 0$, and consequently only half of the contact area needs to be discretized. To apply the MUSE method, the surface region between $x = 0$ to $x = a$ is divided into 10 elements (each being an infinite strip); over each the heat flux is uniform and at the center of each the prescribed temperature is T_c. See Fig. 2.

The influence functions for the geometry shown in this figure can be evaluated with the exact closed-form solution for the problem of a semi-infinite body heated with a constant heat flux over an infinite strip[5] by applying simple superposition. In the MUSE solution presented here, both local and average values of the influence functions[5] are used and the results compared.

Averaged Interface Heat Flux

The spatial average of the interface heat flux can be obtained by summing the products of the elemental heat flux and the fraction of the total contact area occupied by the element

$$\bar{q}(t_M) = \sum_{j=1}^{N} q_{jM}\left(\frac{A_j}{A_c}\right) \tag{8}$$

where q_{jM} is the heat flux at element j at time t_M. For the case in which all elements are infinite strips, Eq. (8) can be written as

$$\bar{q}(t_M) = \sum_{j=1}^{N} q_{jM}\frac{(a_j - a_{j-1})}{a} \tag{9}$$

where a_j, a_{j-1}, and a are shown in Fig. 2. Furthermore, if the elements are equally spaced, the average heat flux can be given by

$$\bar{q}(t_M) = \frac{1}{N}\sum_{j=1}^{N} q_{jM} \tag{10}$$

Approximate Analytical Solution (Single Node)

By considering only one element along the interface, an approximate analytical solution for the average interface heat flux can be obtained by using the Laplace transform technique. This method was first used by Keltner and Beck.[6] Starting with Duhamel's integral for the average interface temperature, one can write

$$T_c - T_0 = \frac{\partial}{\partial t}\int_0^t \bar{q}(\lambda)\bar{\phi}(0, t-\lambda)\,\mathrm{d}\lambda \tag{11}$$

where the influence function $\bar{\phi}(0, t)$ is the spatial average temperature rise over the element ($z = 0$) for a unit heat flux. Taking the Laplace transform of Eq. (11) gives

$$(T_c - T_0) = s\mathscr{L}\left[\int_0^t \bar{q}(\lambda)\bar{\phi}(0, t-\lambda)\,\mathrm{d}\lambda\right] \tag{12}$$

or

$$1/s(T_c - T_0) = s\hat{\bar{q}}(s)\cdot\hat{\bar{\phi}}(s) \tag{13}$$

where the functions $\hat{\bar{q}}(s)$ and $\hat{\bar{\phi}}(s)$ are the transforms of $\bar{q}(t)$ and $\bar{\phi}(t)$, respectively. Solving for $\hat{\bar{q}}(s)$ provides

$$\hat{\bar{q}}(s) = \frac{T_c - T_0}{s^2\cdot\hat{\bar{\phi}}(s)} \tag{14}$$

This equation can be written in dimensionless form as

$$\widehat{\overline{q^+}}(s^+) = \frac{1}{(s^+)^2\cdot\widehat{\overline{\phi^+}}(s^+)} \tag{15}$$

where

$$t^+ = \frac{\alpha t}{a^2},\quad \overline{\phi^+} = \frac{k\cdot\bar{\phi}}{a},\quad s^+ = \frac{sa^2}{\alpha},\quad \overline{q^+} = \frac{\bar{q}\cdot a}{k(T_c - T_0)} \tag{16}$$

Early Time Solution

For small dimensionless times, the average influence function is given by[5]

$$\overline{\phi^+} \simeq 2\left(\frac{t^+}{\pi}\right)^{\frac{1}{2}} - \frac{t^+}{\pi} + \frac{1}{2}(t^+)^3\left(1 - \frac{9}{2}t^+\right)\exp\left(-\frac{1}{t^+}\right) \tag{17}$$

For an error less than 0.033%, the exponential term can be dropped for $t < 0.3$ and thus

$$\overline{\phi^+} \simeq 2\left(\frac{t^+}{\pi}\right)^{\frac{1}{2}} - \frac{t^+}{\pi} \tag{18}$$

Taking the Laplace transform of Eq. (18) yields

$$\widehat{\overline{q^+}} = (s^+)^{-\frac{3}{2}} - \frac{1}{\pi(s^+)^2} \tag{19}$$

Substituting Eq. (19) into Eq. (15) and taking the inverse transform give

$$\overline{q^+}(t^+) = (\pi t^+)^{-\frac{1}{2}} + \frac{1}{\pi}\operatorname{erfc}\left(\frac{-t^{+\frac{1}{2}}}{\pi}\right)\exp\left(\frac{t^+}{\pi^2}\right) \tag{20}$$

This equation can further be simplified to

$$\overline{q^+}(t^+) = (\pi t^+)^{-\frac{1}{2}} + \frac{1}{\pi} \tag{21}$$

which provides less than 3% error for $t^+ < 0.01$.

Late Time Solution

For late dimensionless times, the influence function is given by Eq. 5:

$$\overline{\phi^+} \simeq \frac{1}{\pi}\left(\ell n\, t^+ - \frac{4}{3}\frac{1}{t^+} + 3 - \gamma\right) \qquad (\gamma = 0.5772\ldots) \tag{22}$$

For an error less than 0.2%, the second term on the right-hand side of Eq. (22) can be dropped for $t^+ > 100$, and one can write

$$\overline{\phi^+} = \frac{1}{\pi}(\ell n\, t^+ + 3 - \gamma) \tag{23}$$

Taking the Laplace transform of Eq. (23) gives

$$\widehat{\overline{\phi^+}} = \frac{1}{\pi s^+}(3 - 2\gamma - \ell n\, s^+) \tag{24}$$

Substituting Eq. (24) into Eq. (15) and taking the inverse transform by using the approximate method [which is accurate for $f(s)$ functions that vary slowly with $\ell n\, s$] given in Ref. 7 yields

$$\overline{q^+}(t^+) = \pi[\ell n(2t^+) + 3 - 2\gamma]^{-1} \tag{25}$$

which shows that the average surface heat flux increases as the inverse of the logarithm of time for large times.

Solution to the Interior Region

Once the elemental heat flux histories have been determined, the solution to the interior temperature history $T(x, z, t)$ can be obtained by superimposing the total effect of all these heat fluxes. That is,

$$T(x, z, t_M) = T_0 + \sum_{j=1}^{N}\sum_{i=j}^{M} q_{ji}\,\Delta\phi^{(x,z)}_{t,M-i} \tag{26}$$

where

$$\Delta\phi^{(x,z)}_{j,M-i} = \phi^{(x,z)}_{j,M+1-i} - \phi^{(x,z)}_{j,M-i} \tag{27}$$

and the influence function $\phi^{(x,z)}_{jM}$ is the temperature rise at point (x, z) due to a unit heat flux at element j at time t_M. Equation (26) can be written in a more convenient form as

$$T_M(x, z) = MT_0 - \sum_{k=1}^{M-1} T_k(x, z) + \sum_{j=1}^{N}\sum_{i=1}^{M} q_{ji}\phi^{(x,z)}_{j,M+1-i} \tag{28}$$

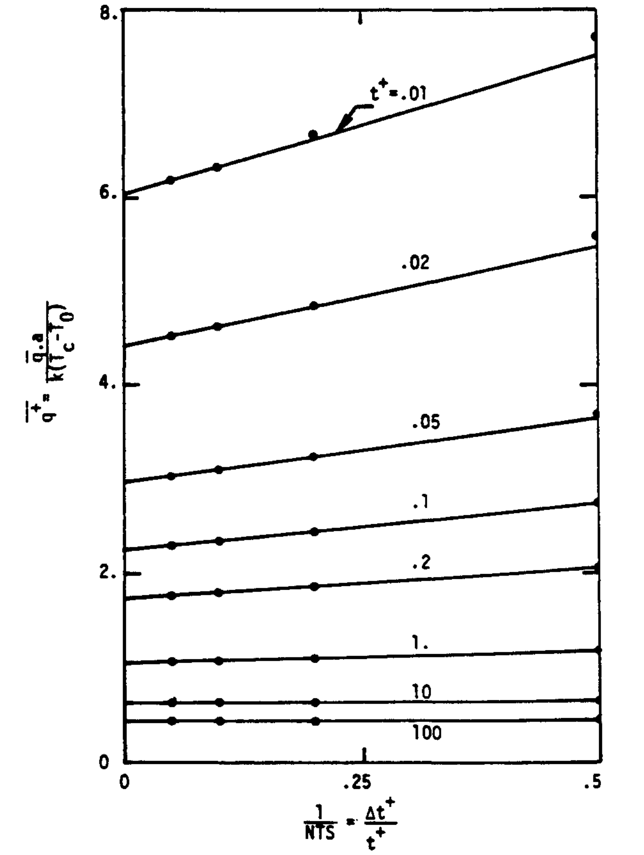

Fig. 3 Variation of the normalized area-averaged interface heat flux with the number of time steps used in the computations.

The solution for the influence function can be obtained from Eq. (24) given in Ref. 5.

Results and Discussion of Accuracy

Two cases of equally spaced elements and variable-spaced elements are examined. The first case is used to find out how accuracy varies with the size of an element, whereas the second case is used to obtain accurate results in the corner region near $x = \pm a$. For both cases, elemental surface heat fluxes are determined for various values of dimensionless time, ranging between $t^+ = 0.01$ to 1000. At each time, the elemental heat fluxes are evaluated in 20 time steps. This means that for larger times, larger time steps are considered. For instance, to determine $q_j(t^+)$ at dimensionless times of 0.01, 1, and 1000, the time steps of 0.0005, 0.05, and 50, respectively, are used.

$$\mathrm{NTS} = \frac{0.01}{0.0005} = \frac{1}{0.05} = \frac{1000}{50} = 20 \tag{29}$$

This substantially reduces the computational effort and provides more uniform accuracies for the results (with respect to NTS) than the case is which a small constant time step is used for the entire time range.

To show how accuracies of the results change with NTS, the case of equally spaced elements is also examined with values of NTS being equal to 20, 10, 5, and 2. In Fig. 3, the normalized area-averaged interface heat flux is plotted vs 1/NTS for different values of dimensionless times. The study of this figure leads to the following observations:

1) The accuracies of the heat flux histories vary linearly with 1/NTS and become more accurate as NTS increases. (The most accurate values can be obtained at 1/NTS = 0 by employing linear extrapolation.) For NTS = 2, however, slight deviation (from a straight line) can be seen at early times, $t^+ < 0.1$. The deviation is less than 2% at $t^+ = 0.01$ and approaches zero as t^+ becomes larger.

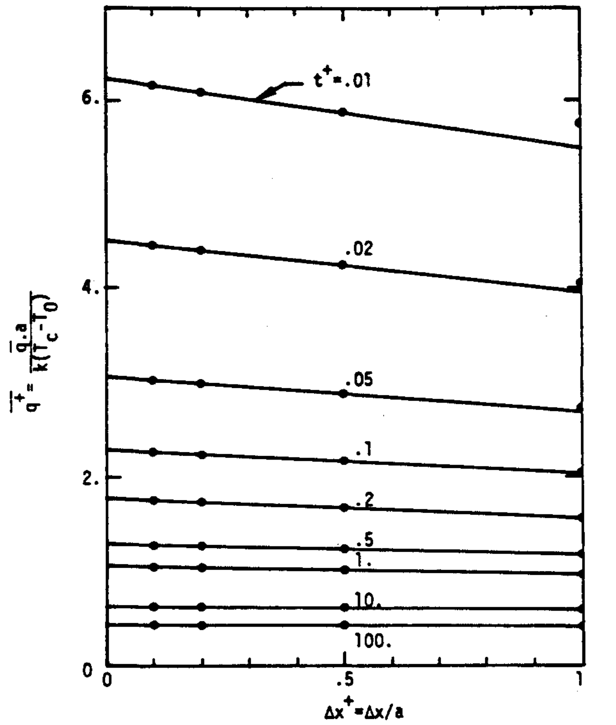

Fig. 4 Variation of the normalized area-averaged surface heat flux with the size of elements.

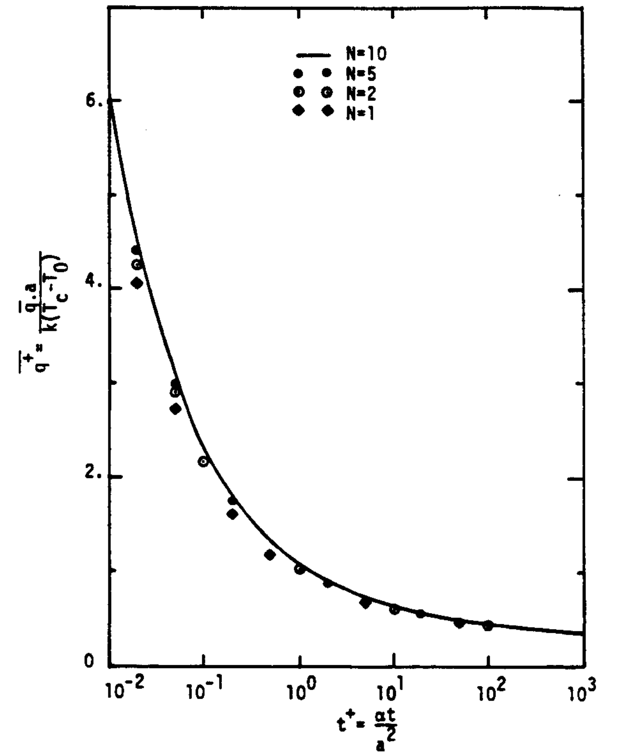

Fig. 5 Results of the average interface heat flux for a different number of elements along the interface (local ϕ used).

2) The slopes of the straight lines shown in the figure are large at early times and become smaller for late times. This implies that, in order to have good accuracy at early times, the NTS should be large, whereas the same accuracy can be obtained with smaller values of NTS at later times. For instance, the required NTS for less than 3% errors in heat flux histories are 20, 10, and 5 for dimensionless times of 0.01, 1, and 10, respectively.

To show how accuracy changes with the size of elements, the case of equally spaced elements is considered with a

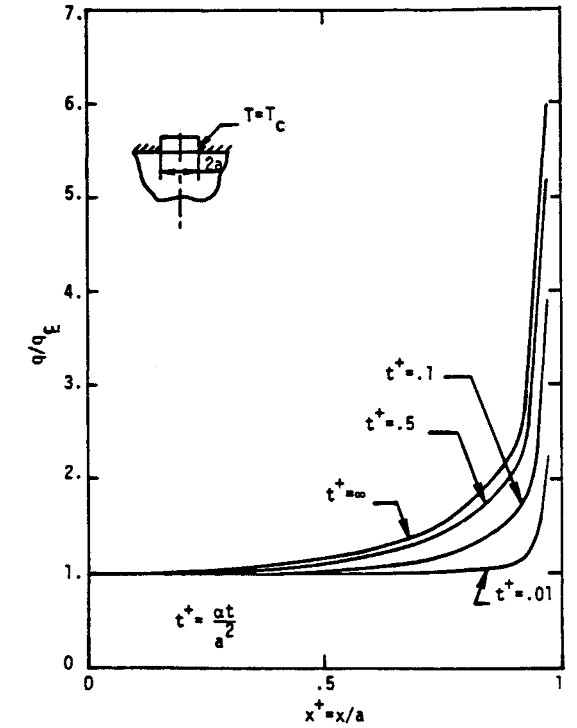

Fig. 6 Normalized heat flux distribution across the strip at various values of time.

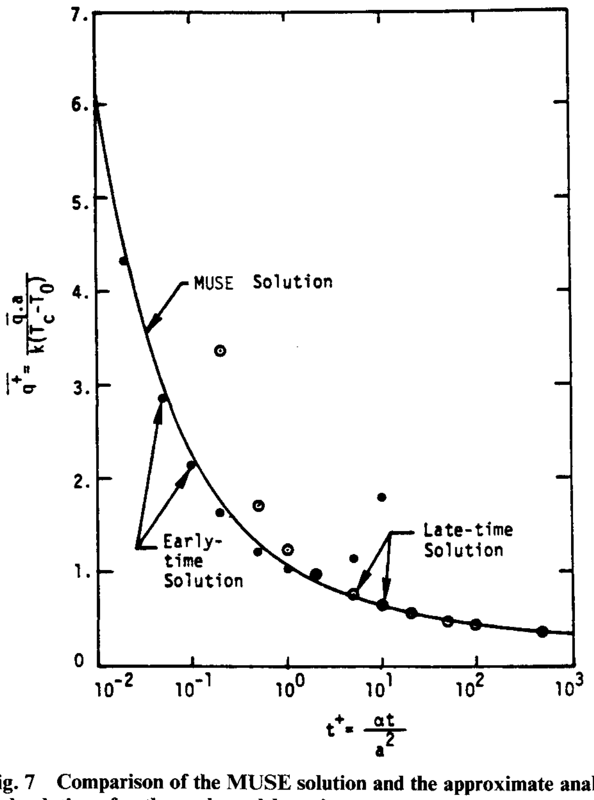

Fig. 7 Comparison of the MUSE solution and the approximate analytical solutions for the early and late times.

different number of elements along the interface ($N = 1$, 2, 5, 10). The number of time steps used was fixed and equal to 20. The results are shown in Figs. 4 and 5. Figure 4 shows the normalized interface heat flux $\overline{q^+}$ vs $\Delta x^+ = \Delta x/a$ for various values of t^+, while in Fig. 5, $\overline{q^+}$ is plotted vs t^+ for different cases of $N = 1$, 2, 5, and 10. A behavior similar to that of Fig. 3 is observed in Fig. 4. It can be seen that $\overline{q^+}$ varies linearly with Δx^+ and becomes more accurate as Δx^+ approaches zero, especially for early times.

Table 1 Comparison between the results obtained with the local and average ϕ solutions. $\overline{q^+} = \bar{q} \cdot a/k(T_c - T_0)$

	Local ϕ solution		Average ϕ solution		Corrected solution
t^+	$\overline{q^+}$	% Error	$\overline{q^+}$	% Error	$N = (\infty)$
0.01	6.1886	1.31	6.2228	0.77	6.2710
0.05	3.0424	1.40	3.0578	0.90	3.0856
0.1	2.2750	1.34	2.2858	0.87	2.3058
0.5	1.3028	1.09	1.3076	0.73	1.3172
1.0	1.0582	0.95	1.0616	0.64	1.0684
10.0	0.6340	0.63	0.6354	0.41	0.6380
100.0	0.4416	0.45	0.4428	0.18	0.4436

The slopes of the lines decrease as t^+ goes to infinity. This indicates that even few elements along the interface can produce good accuracy for large times. (This can also be observed in Fig. 5.)

In order to show precisely the heat flux distribution across the strip, especially in the corner region near $x = \pm a$, 10 variable-spaced elements were used with NTS = 20. The elements near the corner were smaller (about 1/4) than those close to the centerline. Figure 6 shows the normalized spatial variation of the heat flux across the strip at several times. Normalization is obtained by dividing elemental values by the value of the centerline element that covers the region $0 < x^+ 0.2$. It can be seen that the region of uniform heat flux shrinks as t^+ increases. At a dimensionless time about 30, the normalized heat flux distribution remains constant, which indicates that for large times, the heat flux across the strip can be approximated by a product of a function of t^+ and a function of x^+.

As mentioned earlier, both local and average values of the influence functions are examined in this problem. Table 1 shows the results obtained from the local ϕ solution and those found from the average ϕ solution. Both solutions use 10 equally spaced elements with NTS = 20. The first column is the dimensionless time ranging between 0.01 to 100. The second and fourth columns show the normalized values of the area-averaged interface heat flux resulting from the local and average solutions, respectively. The last column provides the corrected values of the averaged interface heat flux for the case in which $N \to \infty$ (corresponding to the values of $\overline{q^+}$ at $\Delta x^+ = 0$ in Fig. 4). Comparison of the results from the second and fourth columns with the corrected values in the last column indicates that the average ϕ solution gives more accurate results than the local ϕ solution. As can be seen in the third and fifth columns, the errors associated with the average solution are about two-thirds of those related to the local solution, which implies that the former solution is more appropriate than the latter one, particularly, when the number of elements is small.

In Fig. 7, the results obtained from the multinode solution (with 10 equally spaced elements and NTS = 20) are compared with those evaluated by the early- and late-time analytical solutions given by Eqs. (20) and (25), respectively. As can be seen, the multinode USE solution is in a very good agreement with the early-time solution up to dimensionless time about 1 and matches closely the late-time solution for times greater than 5.

Conclusions

An unsteady surface-element solution is presented for the problem of a semi-infinite body (with mixed boundary conditions of a step change of the surface temperature over an infinite strip and insulated elsewhere) that is very difficult to solve analytically because of the mixed boundary conditions. The solution is given in terms of the elemental surface heat fluxes that are averaged spatially over the interface. The results were compared with those obtained by the approximate analytical solutions (developed in this paper) for the short and late times. It was found that a very high accuracy is attainable with a relatively small number of surface elements for a long range of dimensionless time.

The accuracy of the results was also examined for the early and late times by having different sizes of elements and a different number of time steps (NTS). It was found that in order to have good accuracy at early times, more elements and larger values of NTS are needed whereas the same accuracy can be obtained at later times with smaller values of NTS and a fewer number of elements along the interface.

Finally, the accuracy of the method was examined by considering both local and average values of the influence functions. It was found that although both approaches produce very accurate results, the errors associated with the average ϕ solution are about two-thirds of those related to the local ϕ solution.

Acknowledgments

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References

[1]Beck, J. V. and Keltner, N. R., "Transient Thermal Contact of Two Semi-Infinite Bodies over a Circular Area," *Progress in Astronautics and Aeronautics: Spacecraft Radiative Transfer and Temperature Control*, Vol. 33, edited by P. L. Bargellini, AIAA, New York, pp. 66–82; also, AIAA Paper 81-1162.

[2]Litkouhi, B. and Beck, J. V., "Multinode Unsteady Surface Element Method with Application to Contact Conductance Problem," *Journal of Heat Transfer*, Vol. 108, May 1986, pp. 257–263.

[3]Sadhel, S. S., "Unsteady Heat Flow Between Solids with Partially Contacting Interface," *Journal of Heat Transfer*, Vol. 103, Feb. 1981, pp. 33–35.

[4]Beck, J. V., "Green's Function Solution for Transient Heat Conduction Problems," *International Journal of Heat and Mass Transfer*, Vol. 27, 1984, pp. 1235–1244.

[5]Litkouhi, B. and Beck, J. V., "Temperature in Semi-Infinite Body Heated by a Constant Heat Flux over Half Space," *Heat Transfer, 1982 Munchen Conference Proceedings*. Vol. 2, Hemisphere, New York, 1982, pp. 21–27.

[6]Keltner, N. R. and Beck, J. V., "Unsteady Surface Element Method," *Journal of Heat Transfer*, Vol. 103, Nov. 1981, pp. 759–764.

[7]Schapery, R. A., "Approximate Methods of Transform Inversion for Viscoelastic Stress Analysis," *Proceedings of the Fourth U.S. National Congress of Applied Mechanics*, 1961, pp. 1075–1085.